# On properties of families of sets

## Lecture 4

**Lajos Soukup**

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences

**http://www.renyi.hu/∼soukup**

7th Young Set Theory Workshop

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**Definition**

*A poset $\langle P, \leq \rangle$ has the weak Freese-Nation property iff there is $f : P \to [P]^{\omega}$ such that for any $p, q \in P$ if $p \leq q$ then there is $r \in f(p) \cap f(q)$ s.t. $p \leq r \leq q$.*

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**Theorem**

*If $V = L$, then the poset $\langle [\kappa]^{\omega}, \subset\rangle$ has the wFN-property.*

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*GCH does not imply that* $\left\langle \left[\aleph_{\omega}\right]^{\omega}, \subset \right\rangle$ *has the wFN property.*

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- $b_\beta \subset U'$ so there is $b_\zeta \in F(b_\beta) \cap F(U')$ such that $b_\beta \subset b_\zeta \subset U$.
- Then $\zeta \in I$, and so $\zeta < \alpha$. Thus $b_\zeta \cap b_\beta$ is finite by $(\star)$. Contradiction.

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*provided* $\{\kappa : cf([\kappa]^\omega, \subset) = \kappa\}$ *is cofinal in* $\{\kappa < 2^\omega : cf(\kappa) > \omega\}$

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**Theorem (Fuchino, Geschke, Shelah, S, 2001)**

*If V* $\models$ *"GCH and* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$*", and* $\mathbb{H}$ *is the Hechler poset in V adding a dominating real, then*

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Then there is a continuous sequence $\langle A_\zeta : \zeta < \omega \cdot \mathrm{cf}(\lambda)\rangle \subset [\lambda]^{<\lambda}$ s.t.

(a) $A_0 = \emptyset$, $\cup\{A_\zeta : \zeta < \omega \cdot \mathrm{cf}(\lambda)\} = \lambda$,

(b) $\forall \zeta\ A_{\zeta+1}/A_\zeta \in F$.

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# Poor man's singular cardinal compactness

Assume $\lambda > \mathrm{cf}(\lambda)$, $G \subset [\lambda]^{<\lambda}$, $F \subset \{\langle B, A\rangle : A \subset B \subset \lambda\}$.

Then there is a continuous sequence $\langle A_\zeta : \zeta < \omega \cdot \mathrm{cf}(\lambda)\rangle \subset [\lambda]^{<\lambda}$ s.t. $A_0 = \emptyset$, $\cup\{A_\zeta : \zeta < \omega \cdot \mathrm{cf}(\lambda)\} = \lambda$, $\forall \zeta\ \langle A_{\zeta+1}, A_\zeta\rangle \in F$

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(A set system $\mathcal{G}$ is $\rho^*$-chain closed iff $\bigcup_{\alpha<\rho^*} G_\alpha \in \mathcal{G}$ for any increasing sequence $\langle G_\alpha : \alpha < \rho^*\rangle \subset \mathcal{G}$.)

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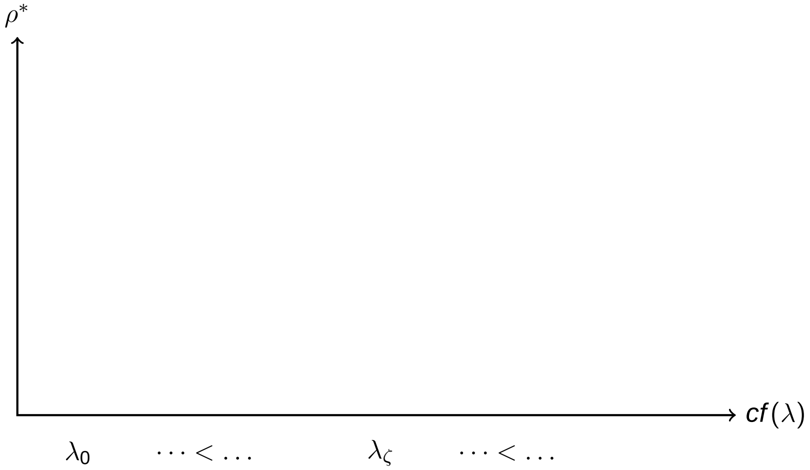

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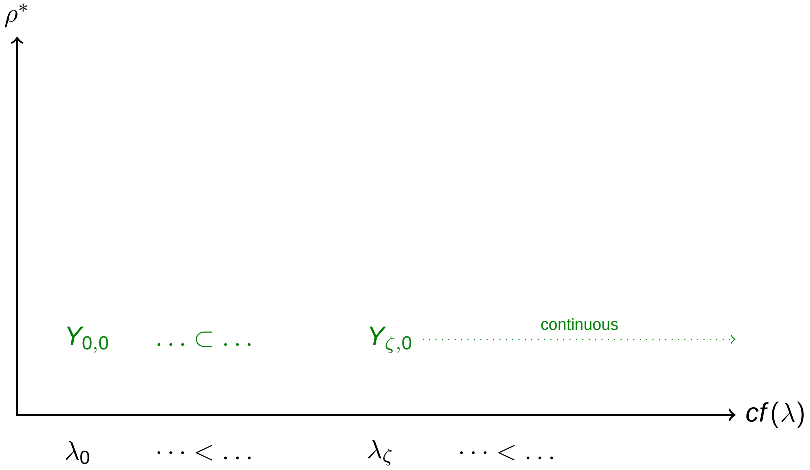

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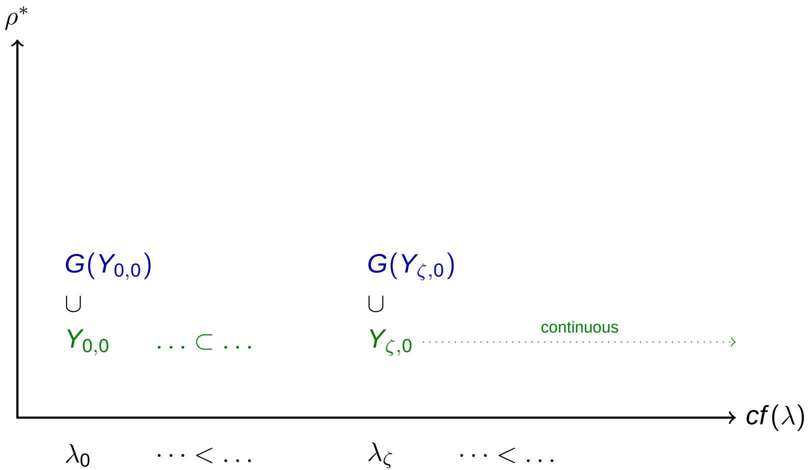

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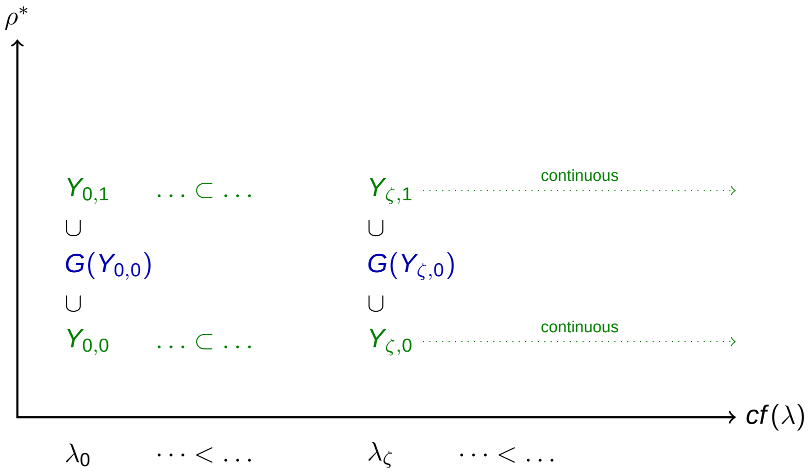

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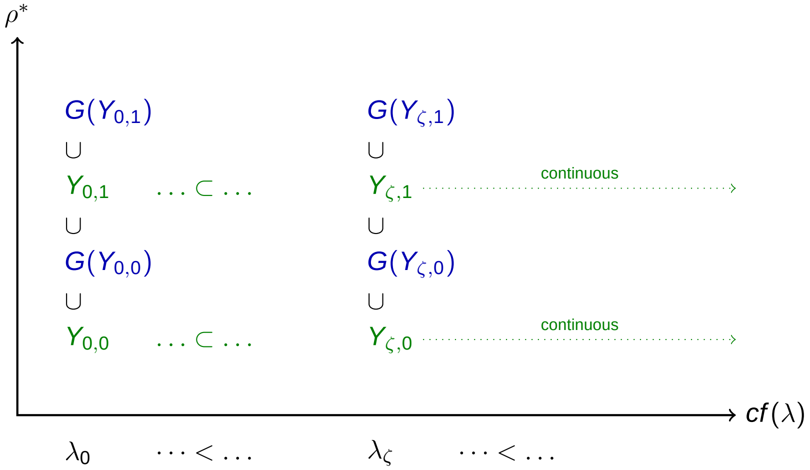

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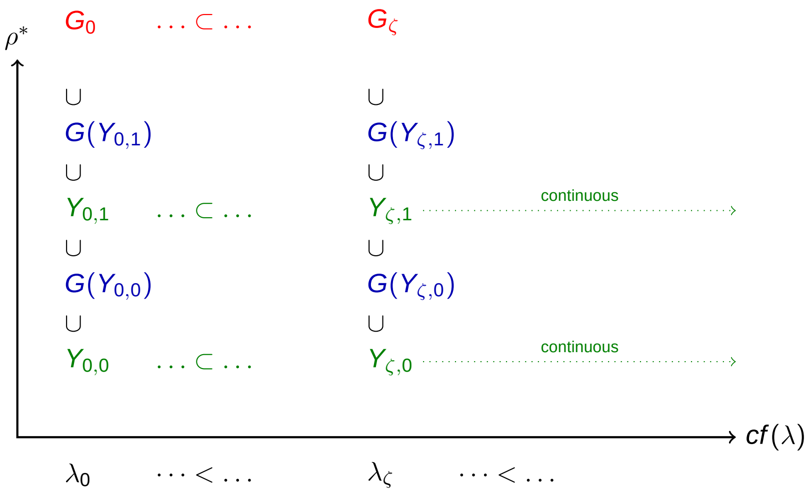

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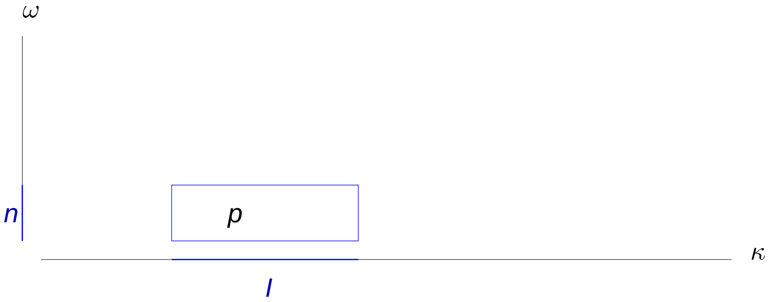

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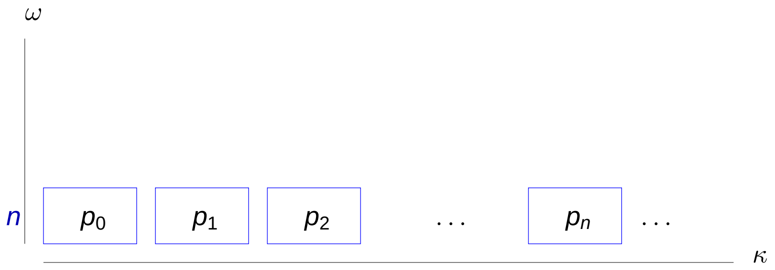

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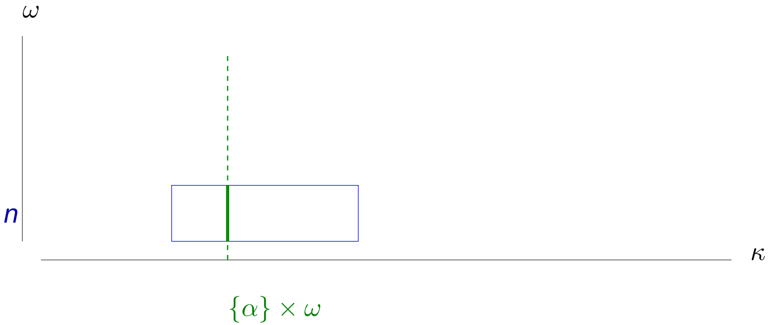

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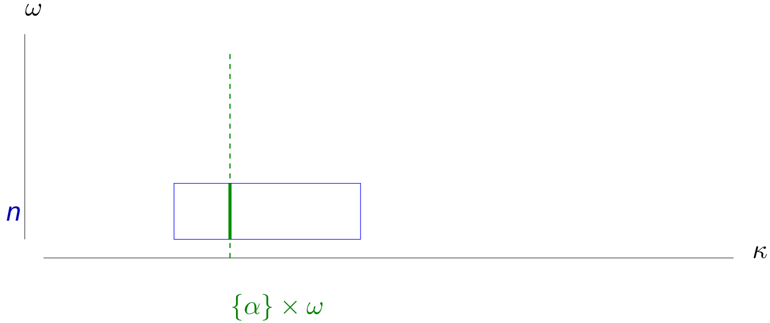

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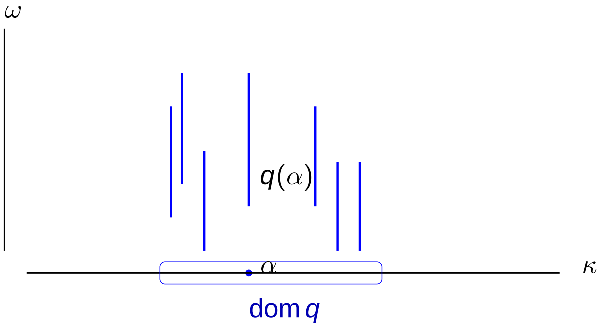

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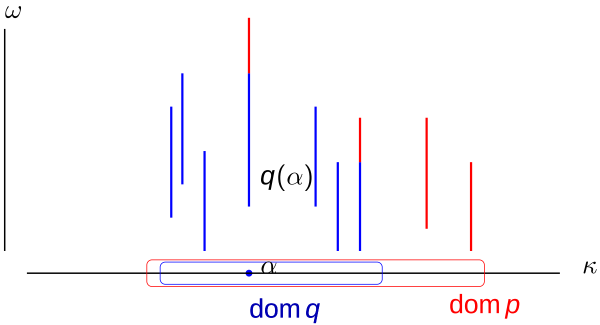

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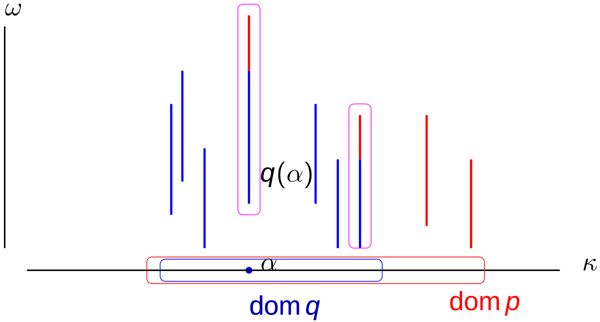

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## Lemma (S)

*If CH holds, $|Q| = \omega_1$ and $Q$ has property K, then forcing with $P_\kappa$ preserves all the cardinals.*

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*It is consistent that*

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- $\exists \mathcal{A} \in \mathbb{A}\ \omega_1 \to [\mathcal{A}]^1_{\omega,<\omega}$ *(i.e.* $\forall c : \omega_1 \to \omega\ \exists A \in \mathcal{A}\ |c[A]| < \omega$*.)*

# Properties of ladder systems on $\omega_1$

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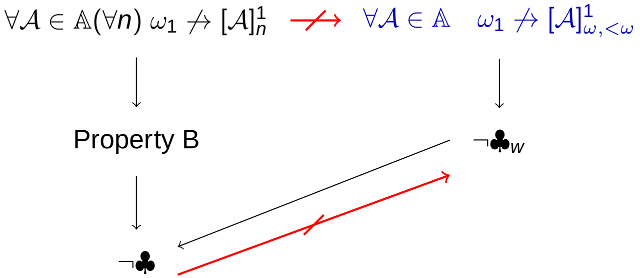

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Theorem (Z. Balogh,2002)

*Assume Axiom R. Then every locally compact $\aleph_1$-metrizable space is metrizable.*

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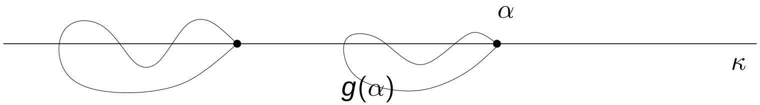

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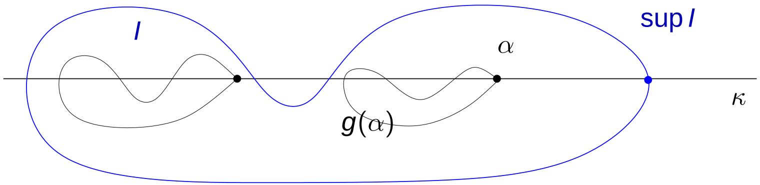

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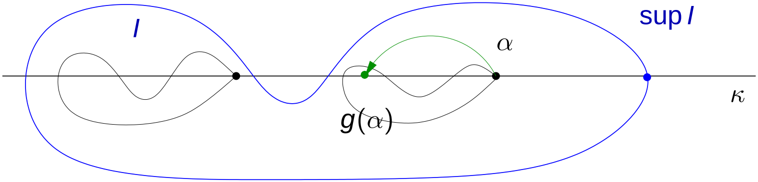

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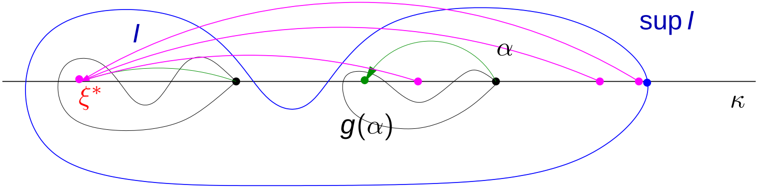

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Theorem (Fuchino, Juhász, S., Szentmiklóssy, Usuba, 2010)
*Assume FRP. Then every locally compact $\aleph_1$-metrizable space is metrizable.*

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- *For any graph G,*
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# The Zoo of the properties of families of sets

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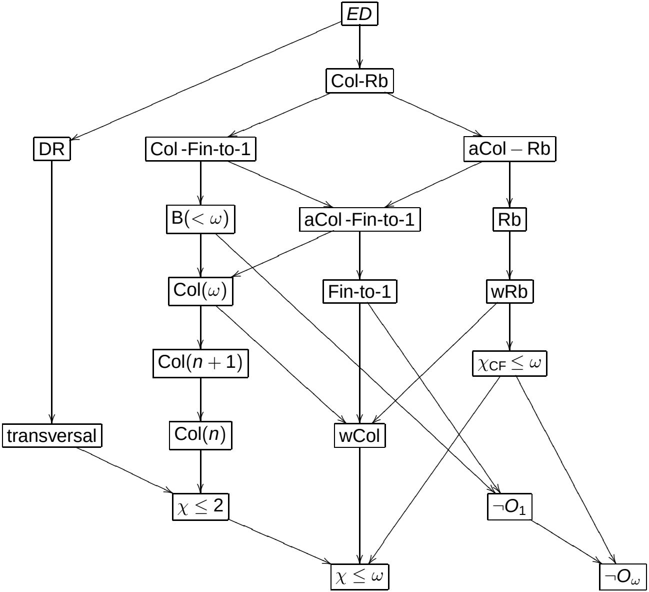

Thank you!

http://www.renyi.hu/∼soukup