# On properties of families of sets
Lecture 2

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7th Young Set Theory Workshop

# Recapitulation

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Definition: A family $\mathcal{A} \subset \mathcal{P}(X)$ has property B iff $\chi(\mathcal{A}) = 2$, where the chromatic number of $\mathcal{A}$ is defined as follows:

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An $n$-almost disjoint family of infinite sets has property $B$.

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Theorem (Davies, 1963)

*$\mathbb{R}^2$ is the union of countably many rotations of functions.*
*If $\alpha_0, \alpha_1, \ldots$ are pairwise different angles between 0 and $\pi$, then there are function $f_0, f_1 \ldots$ such that*

$$\mathbb{R}^2 = \bigcup_{n\in\omega} R_{(\alpha_n)}(f_n).$$

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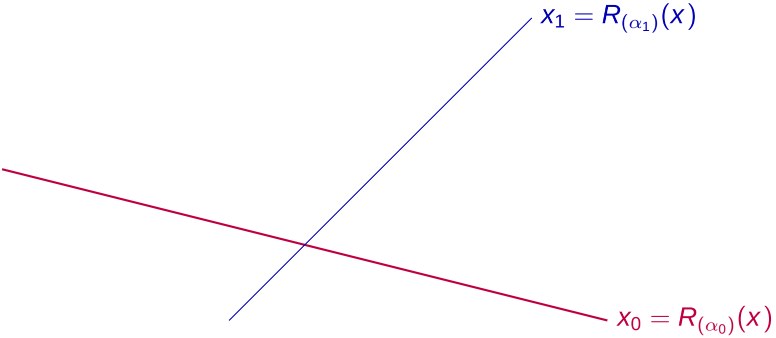

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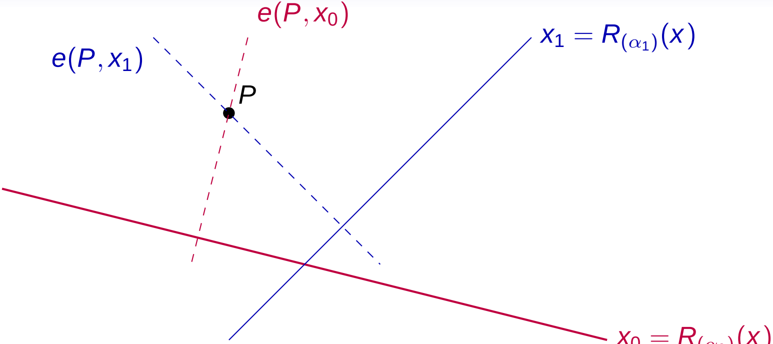

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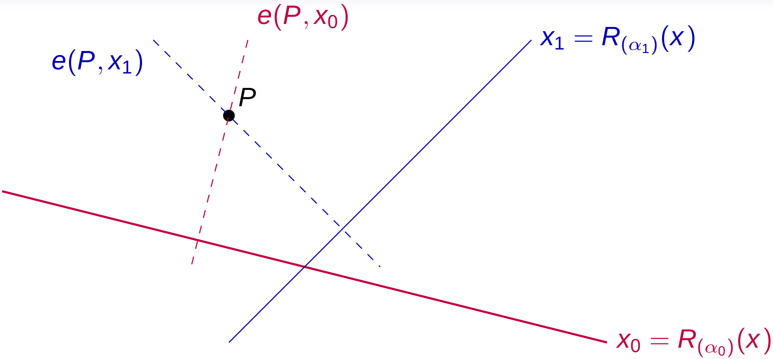


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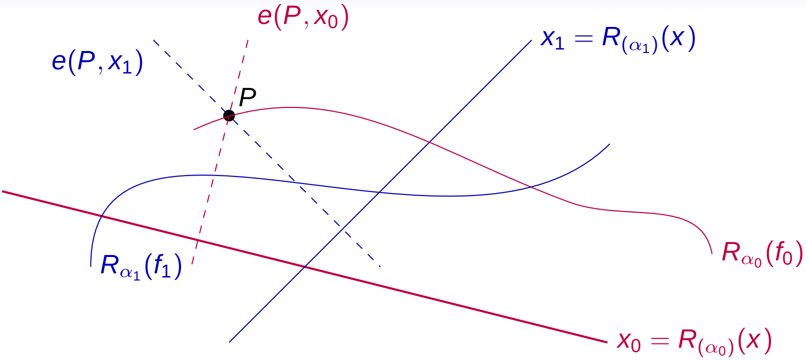


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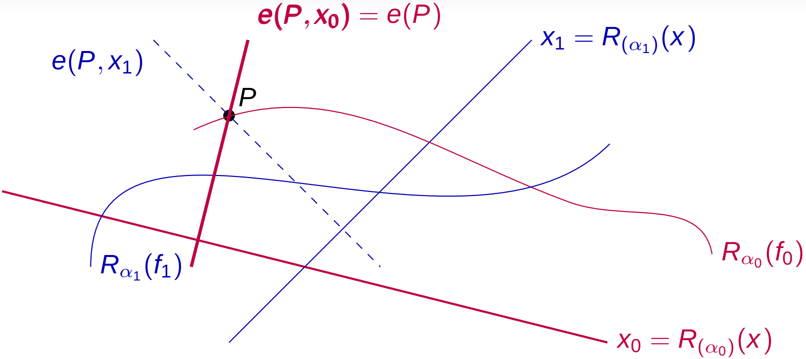


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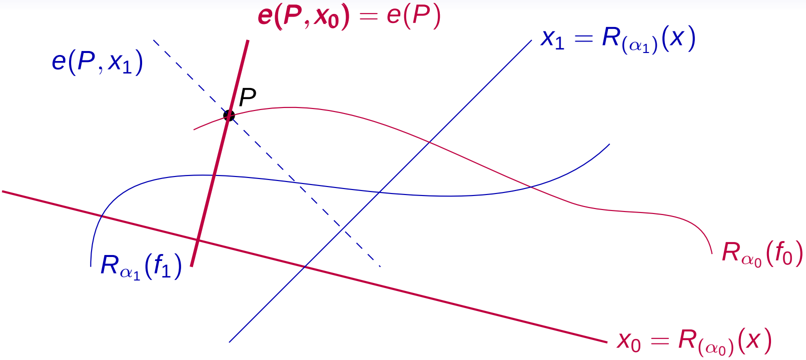


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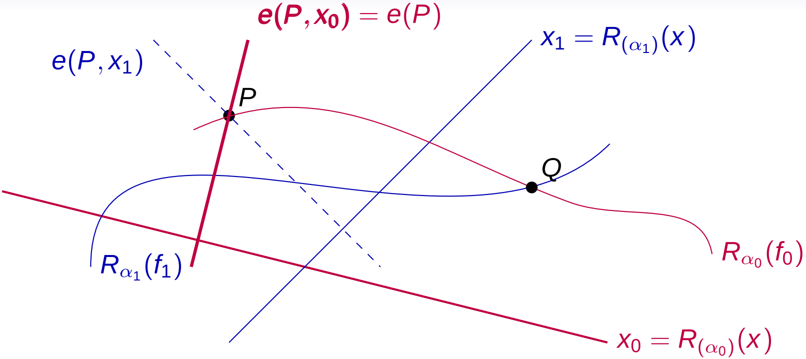


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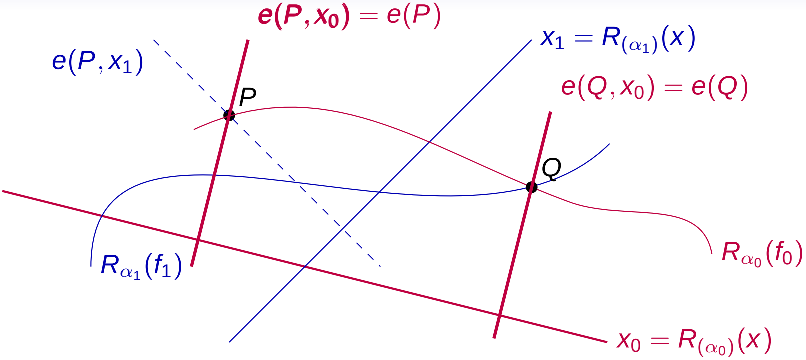


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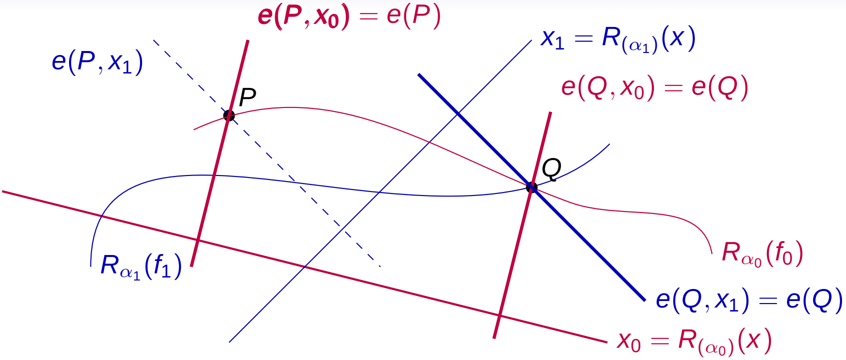


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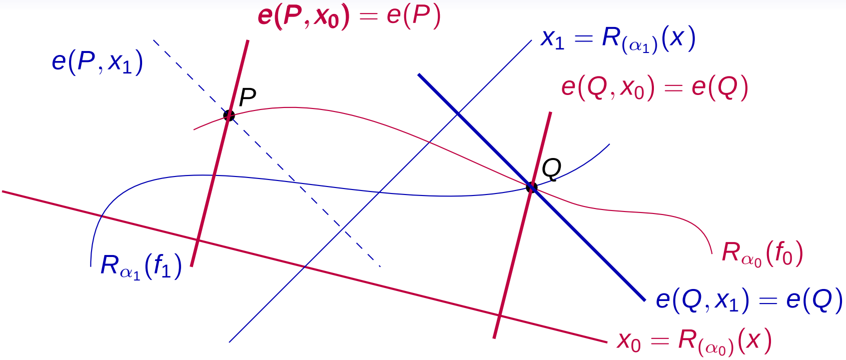


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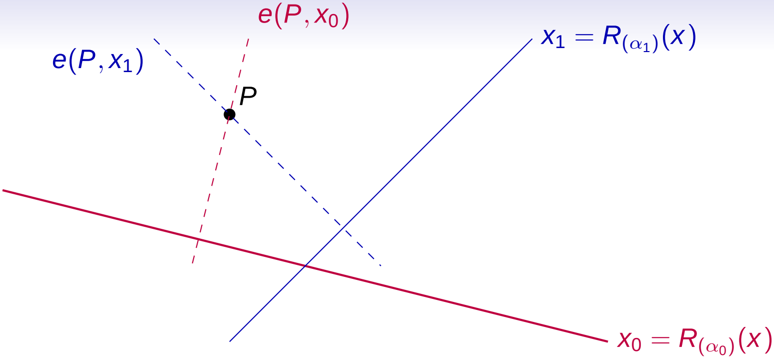
- $\mathbb{E}$ has a transversal, i.e. an injective choice function.

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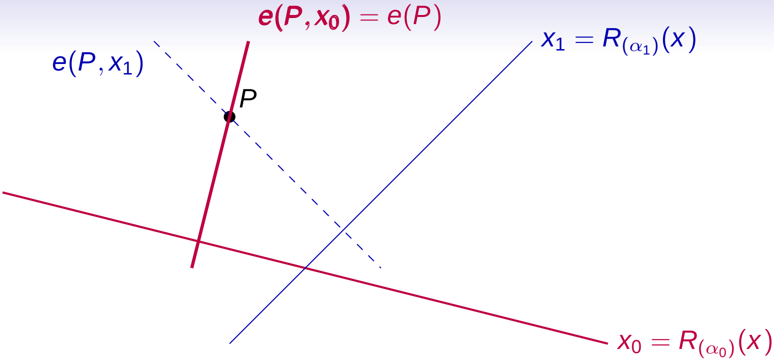


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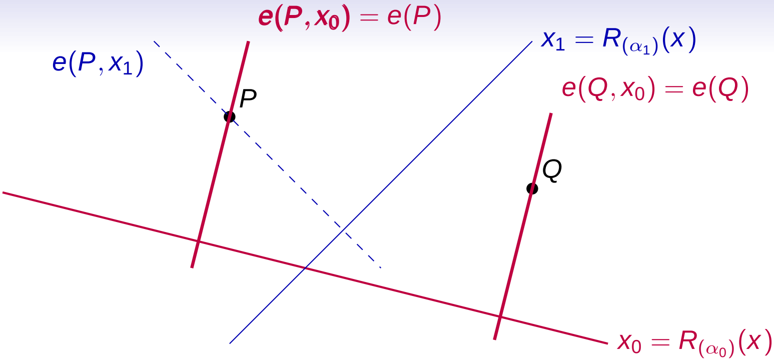


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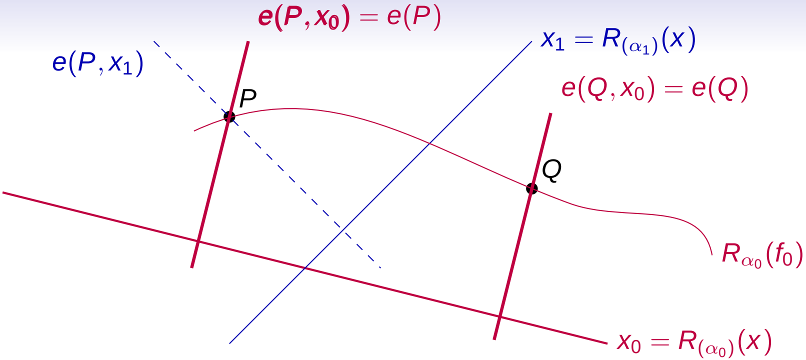


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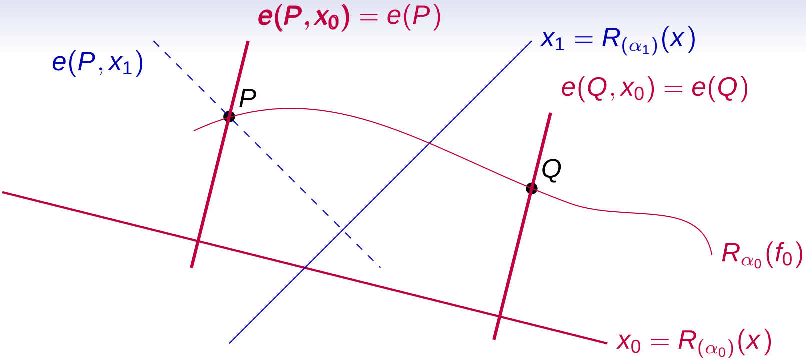


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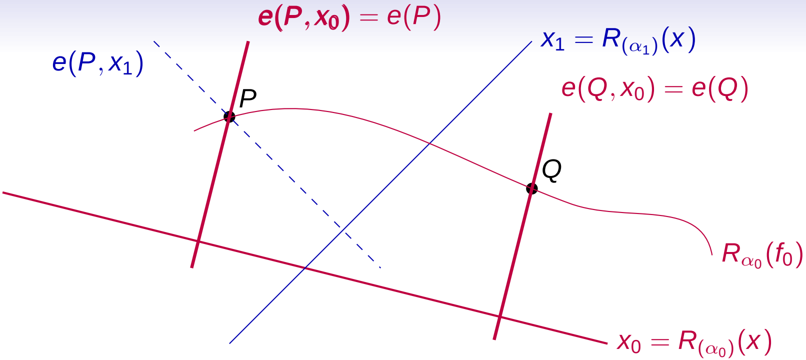


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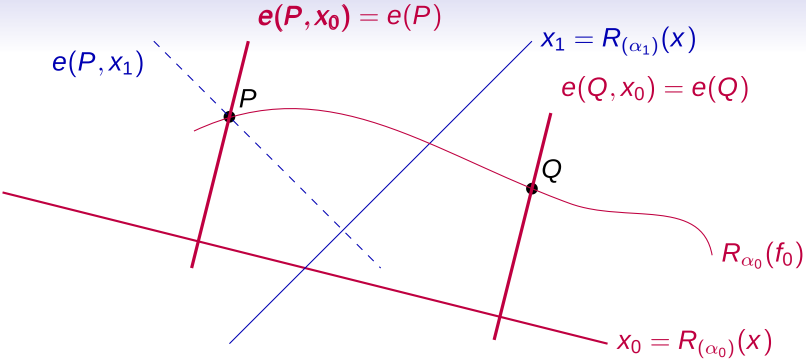


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- $\mathbb{E}$ is 2-almost disjoint.

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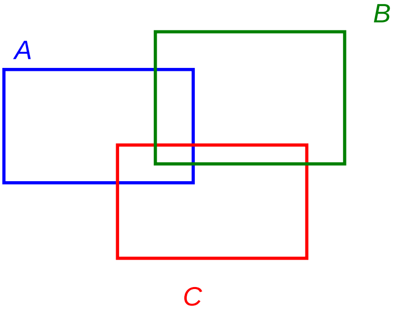

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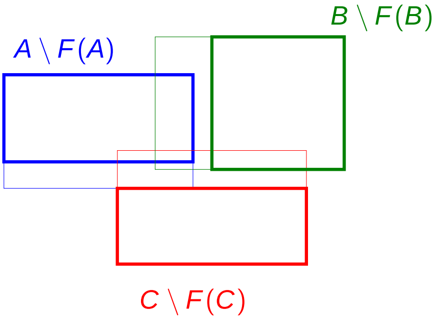

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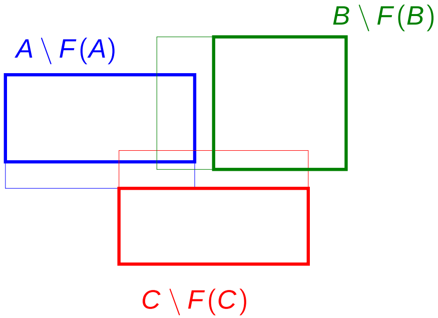

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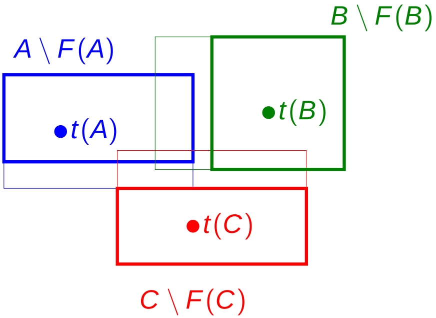

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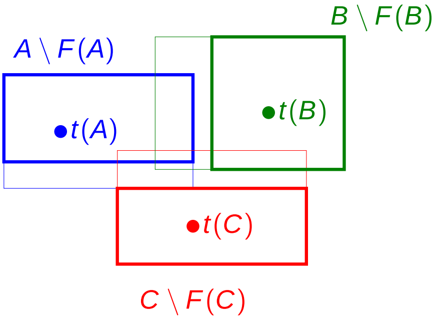


- There is a coloring $c$ such that $c[X] = \omega$ for all $X \in \mathcal{A}$. Enough: $c[X \setminus F(X)] = \omega$.

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- Consider the case $\mathcal{A} = \{A_\zeta : \zeta < \omega_1\} \subset [\omega_1]^\omega$

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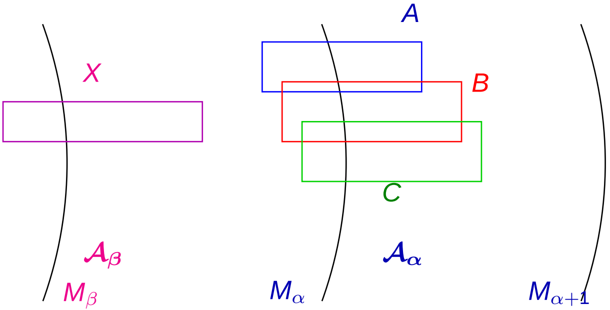

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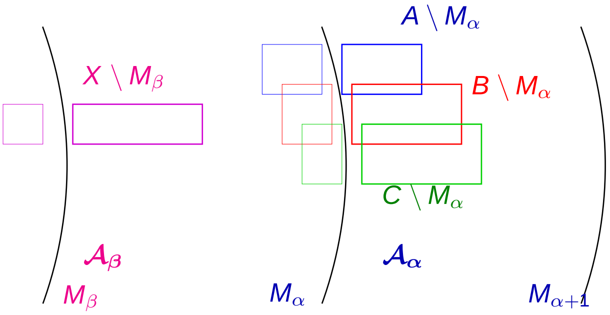

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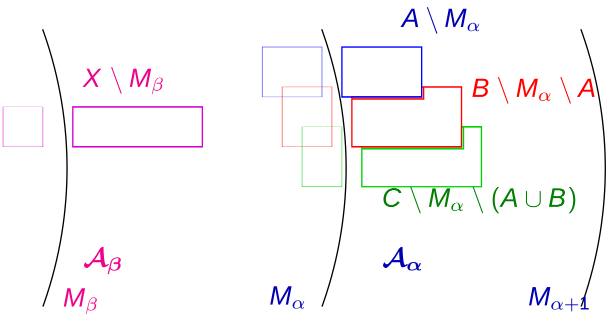

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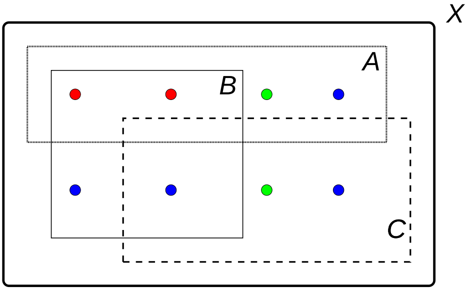

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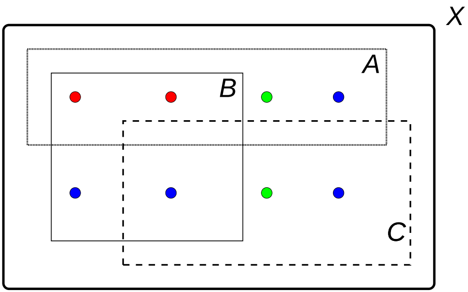

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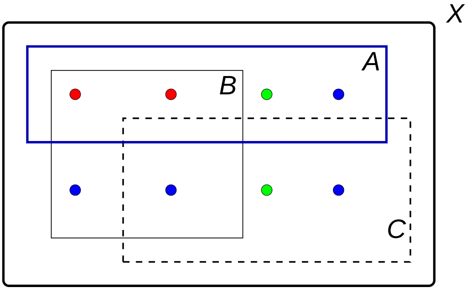

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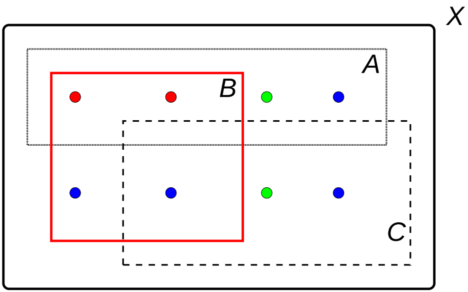

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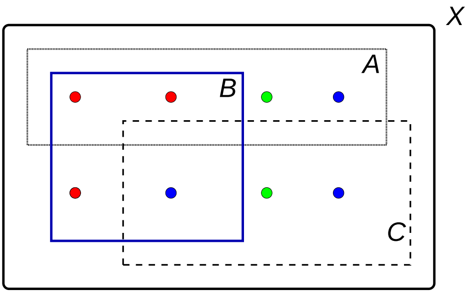

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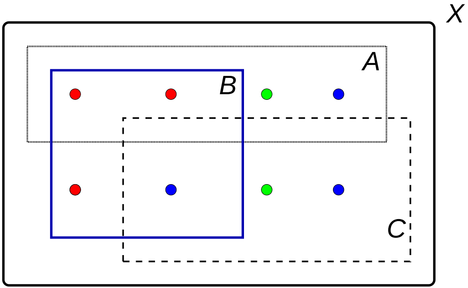

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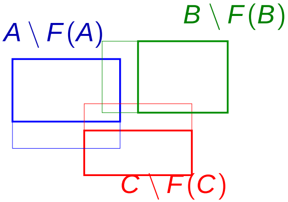

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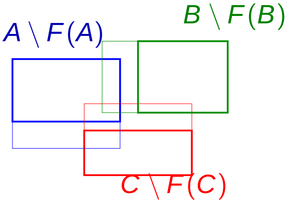


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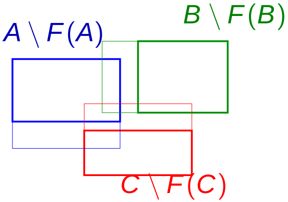


- There is $f : \lambda \to \omega$ such that $f \restriction A \setminus F(A)$ is 1-to-1 for all $A \in \mathcal{A}$.
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# Conflict free colorings of $n$-ad families

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- Define $g : \lambda \to \omega$ s.t. $f \subset g$ and $g(x) = 0$ for all $x \in \lambda \setminus Y$.

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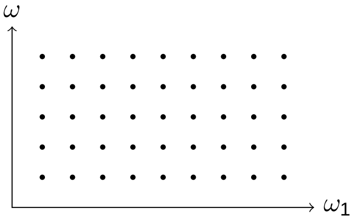

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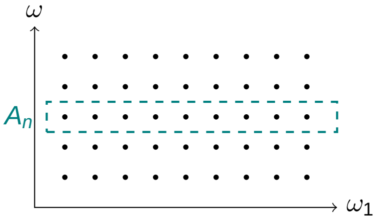


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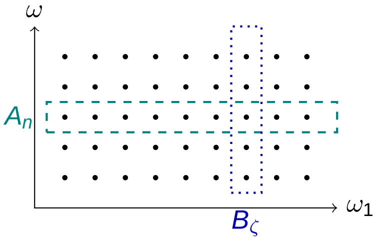


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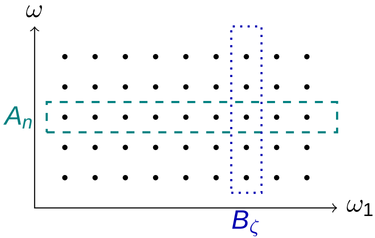


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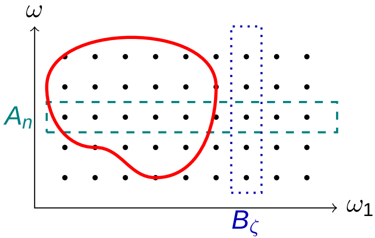


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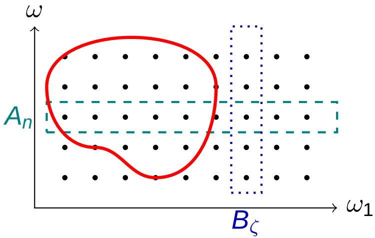


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Theorem (Komjáth, 2012)

$\chi_{CF}(\mathcal{A}) \leq \omega$ *for each n-almost disjoint* $\mathcal{A} \subset [\kappa]^{\geq\omega}$.

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- $\mathcal{A}$ is $\omega_1$-essentially disjoint iff for each $A \in \mathcal{A}$ there is $F(A) \in [A]^{<\omega_1}$ such that $\{A \setminus F(A) : A \in \mathcal{A}\}$ is disjoint.

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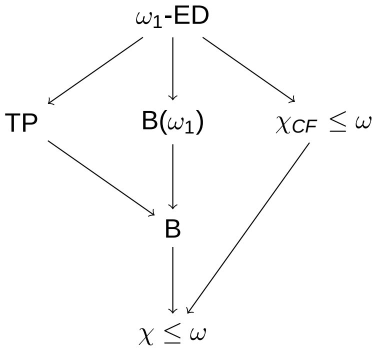

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# Positive results concerning almost disjoint families

**V=L**: $\exists \langle M_\alpha : \alpha < \kappa \rangle$ s.t. ( if $|X \cap M_\alpha| \geq \omega_1$ then $M_\alpha \cap [X]^\omega \neq \emptyset$)

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# Inhomogeneous systems

- every *n*-ad subfamily of $[\lambda]^{\omega}$ is **ED**
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Theorem (S, 2012)

*If every $\mu$-almost disjoint subfamily of $[\lambda]^{\kappa}$ is $\kappa$-**ED**, then $\chi_{\mathrm{CF}}(\mathcal{A}) \le \kappa$ for all $\mu$-almost disjoint $\mathcal{A} \subset [\lambda]^{\ge\kappa}$.*

So the expected results hold.

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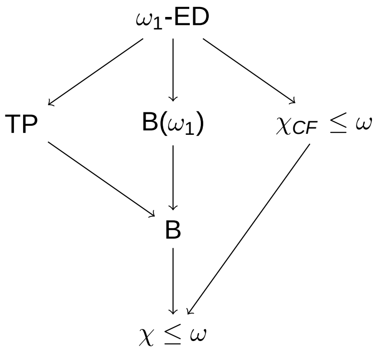

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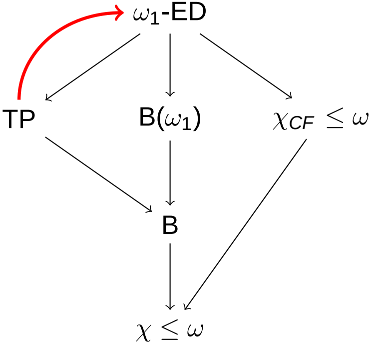

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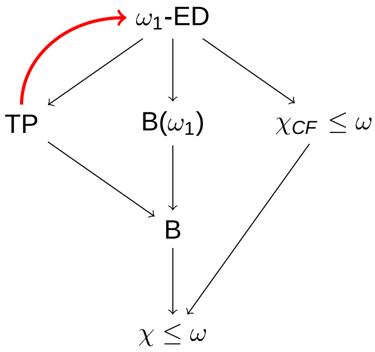


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*If every $\omega$-almost disjoint subfamily of $[\lambda]^{\omega_1}$ has the transversal property*
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Definition

*$\rho^{[\nu]} = \rho$ iff there is a family $\mathcal{B} \subset [\rho]^{\leq \nu}$ of size $\rho$ such that for all $u \in [\rho]^{\nu}$ there is $\mathcal{P} \in [\mathcal{B}]^{<\nu}$ such that $u = \cup \mathcal{P}$.*

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Theorem (Shelah, Revised GCH)

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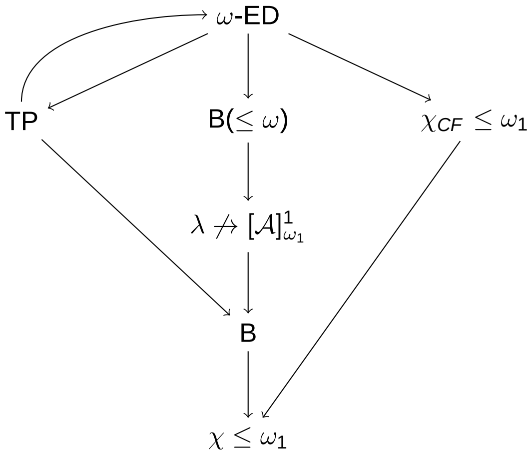

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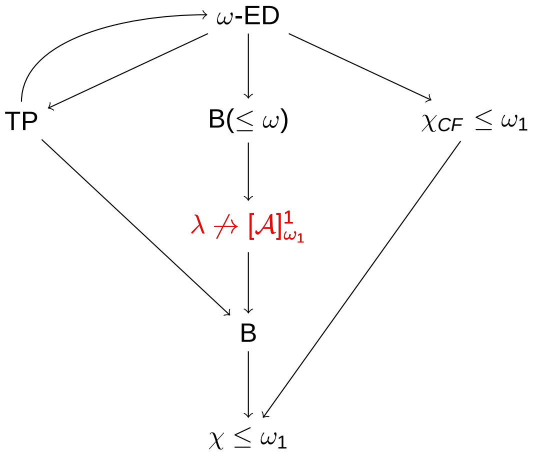

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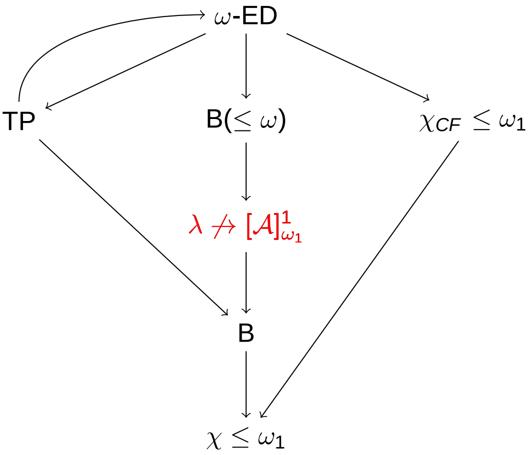


$\lambda \not\to [\mathcal{A}]^1_{\omega_1}$ iff there is $f : \lambda \to \omega_1$ s.t. $f[A] = \omega_1$ for all $A \in \mathcal{A}$.

To be continued . . .