

On properties of families of sets

Lecture 1

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7th Young Set Theory Workshop

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- Miller, Edwin W. **On a property of families of sets**. C. R. Soc. Sci. Varsovie 30, 31-38 (1937).

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- Erdős, Paul; Hajnal, András **On a property of families of sets.** Acta Math. Acad. Sci. Hung. 12, 87-123 (1961).

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$$\exists X \subset \mathbb{R} \quad \forall \text{ perfect } P \quad P \cap X \neq \text{empty} \text{ and } P \setminus X \neq \emptyset$$

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Question:

Which families have property B?

Almost disjoint families

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Theorem (Balcar, Vojtáš, 1980)

Every non-principal ultrafilter \mathcal{U} on ω has an *almost disjoint refinement*, i.e. a family $\mathcal{A}_\mathcal{U} = \{A_U : U \in \mathcal{U}\}$ such that

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Clearly $\chi(\mathcal{A}_\mathcal{U}) = \omega$: if $c : \omega \rightarrow n$, then there is $U \in \mathcal{U}$ with $c[U] = \{i\}$, and so $c[A_U] = \{i\}$.

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- Then $c[A_P] = \{i\}$.

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Theorem (Brendle, Balcar-Pazak, 2008)

If $V \subset W$ are models of ZFC, W contains a new real r , then $V \cap [\omega]^\omega$ has an almost disjoint refinement in W .

Thm: If $V \subset W$ are models of ZFC, W contains a new real r , and $\{x_\alpha : \alpha < \kappa\} = [\omega]^\omega \cap V$, then there is an almost disjoint family $\{a_\alpha : \alpha < \kappa\} \subset [\omega]^\omega$ in W s.t. $a_\alpha \subset x_\alpha$.

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- If $B(x)_y$ contains a perfect set then $|(B(x)_y)^W \setminus V| \geq \kappa = (2^\omega)^V$.

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- Hence $|B(x)_y| \leq \omega$ **OR** $B(x)_y$ contains a perfect set.
- If $B(x)_y$ contains a perfect set then $|B(x)_y^W \setminus V| \geq \kappa = (2^\omega)^V$.
- Construct $\{a_\alpha : \alpha < \kappa\}$ inductively. We will define γ_α, b_α as well

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- $G_\alpha \cap G_\beta \subset \bigcup_{1 \leq i \leq n+1} (F_\alpha^i \cap F_\beta^i)$, and $F_\alpha^i \cap F_\beta^i$ is finite.
- $G_\alpha \subset D_\alpha^j$, so $(D_\alpha^1, \dots, D_\alpha^n)$ can not prove $\chi(\mathcal{G}) \leq n$.

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- $\{(D_\alpha^1, \dots, D_\alpha^n) : \alpha < 2^\omega\}$ partitions of ω into n sets.
- for all $1 \leq i \leq n + 1$ there is $1 \leq j \leq n$ s.t. $F_\alpha^i \cap D_\alpha^j$ is infinite
- there is $1 \leq j \leq n$ and there are $1 \leq i_1 < i_2 \leq n$ such that both $F_\alpha^{i_1} \cap D_\alpha^j$ and $F_\alpha^{i_2} \cap D_\alpha^j$ are infinite.
- Let $G_\alpha = D_\alpha^j \cap (F_\alpha^{i_1} \cup F_\alpha^{i_2})$.

Claim: $\mathcal{G} = \{G_\alpha : \alpha < 2^\omega\}$ is almost disjoint and $\chi(\mathcal{G}) = n + 1$.

- $G_\alpha \cap G_\beta \subset \bigcup_{1 \leq i \leq n+1} (F_\alpha^i \cap F_\beta^i)$, and $F_\alpha^i \cap F_\beta^i$ is finite.
- $G_\alpha \subset D_\alpha^j$, so $(D_\alpha^1, \dots, D_\alpha^n)$ can not prove $\chi(\mathcal{G}) \leq n$.
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Almost disjoint families with prescribed chromatic number

Theorem (Erdős, Shelah, 1972)

$\forall n \geq 2$ there is an almost disjoint family $\mathcal{G} \subset [\omega]^\omega$ with $\chi(\mathcal{G}) = n + 1$.

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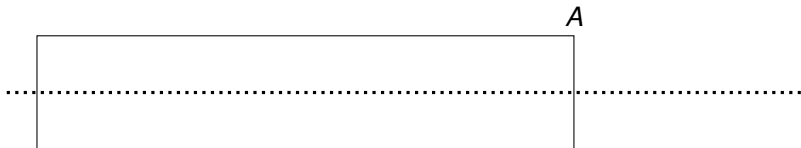
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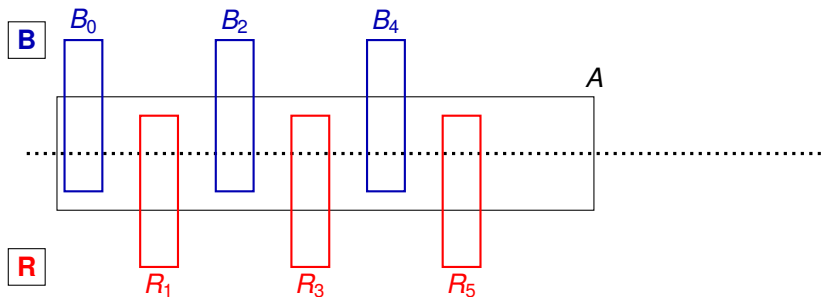
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R

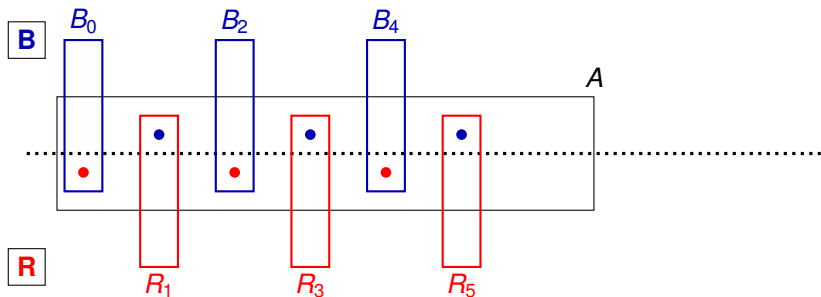
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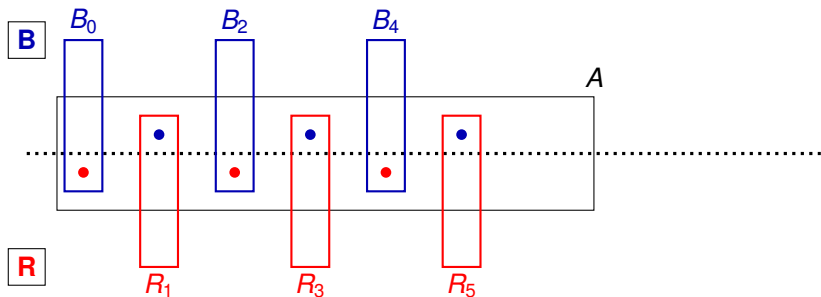
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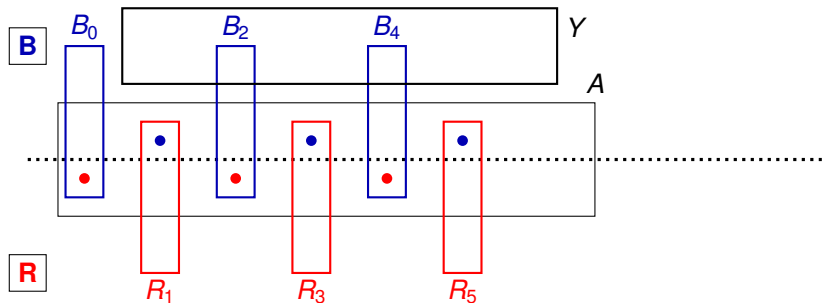
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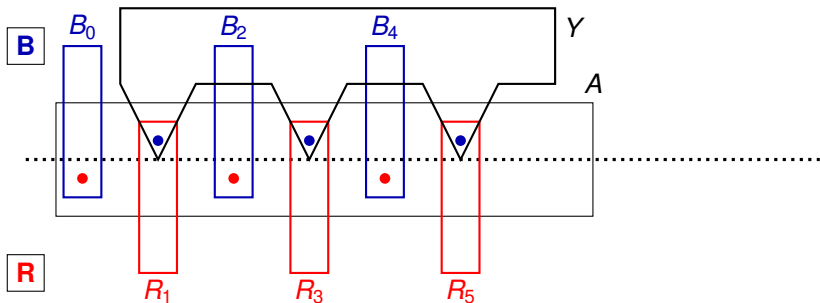
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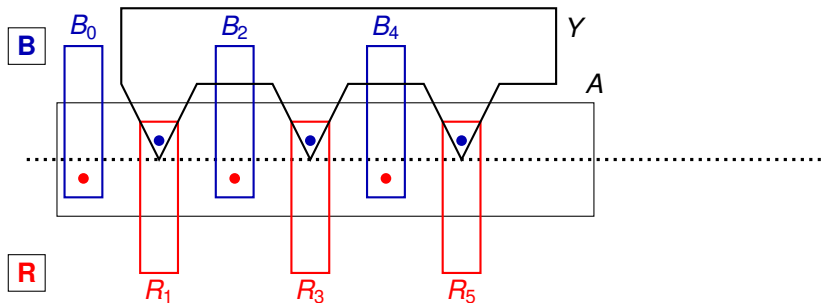
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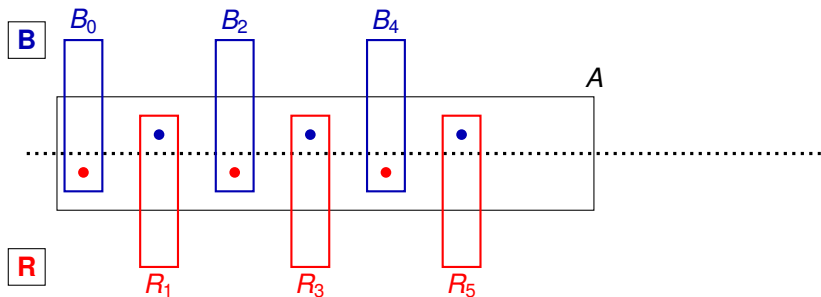
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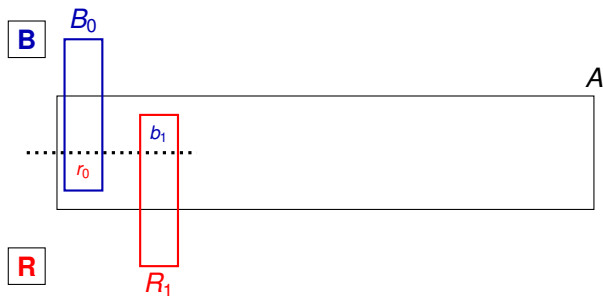
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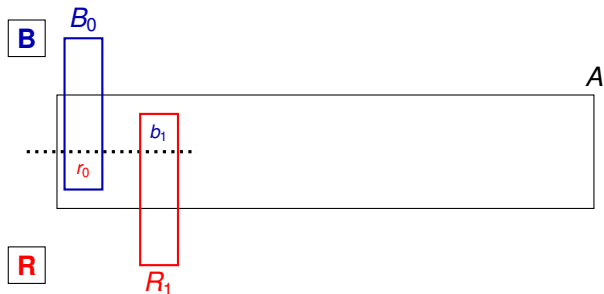
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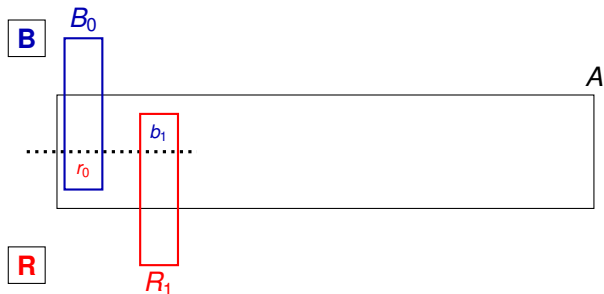
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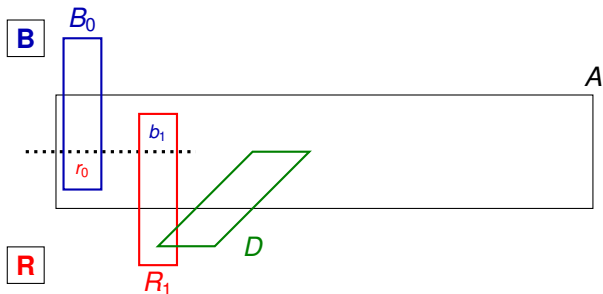
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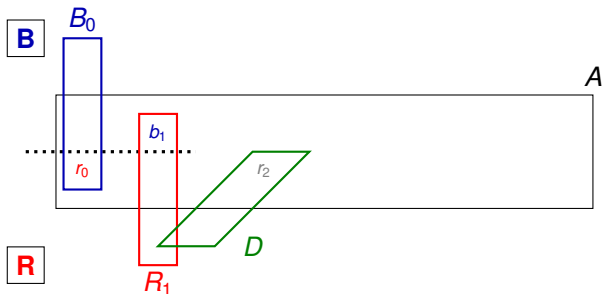
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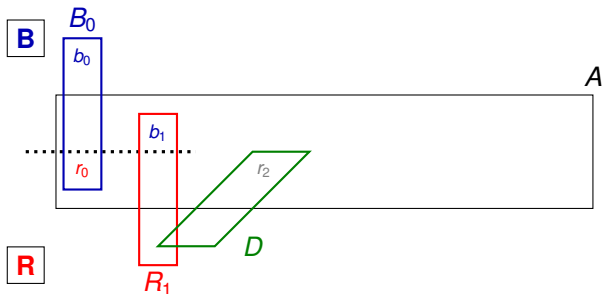
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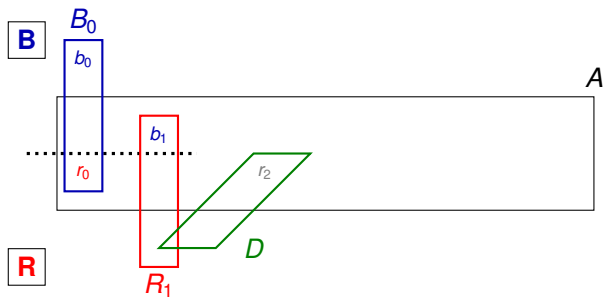
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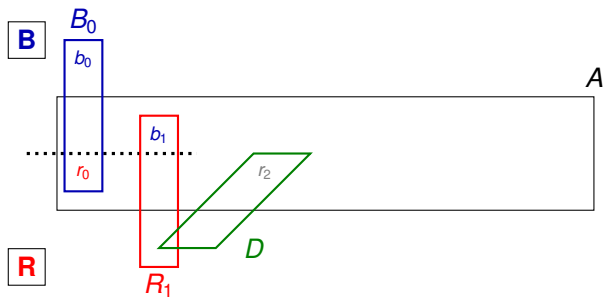
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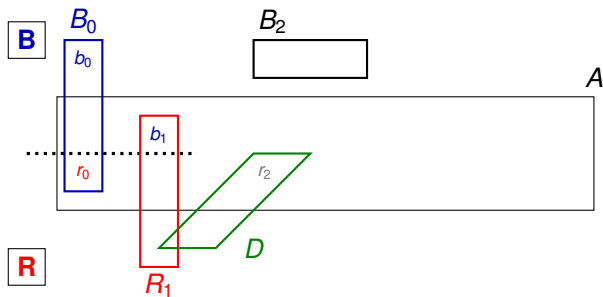
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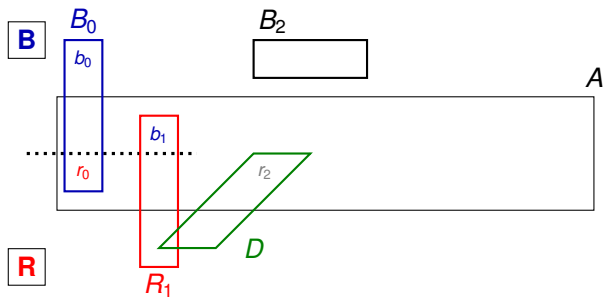
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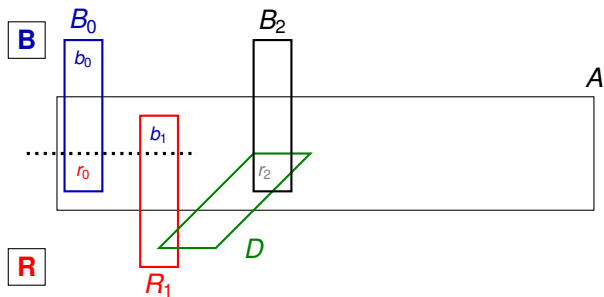
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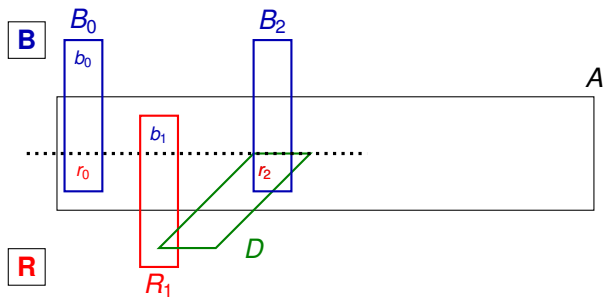
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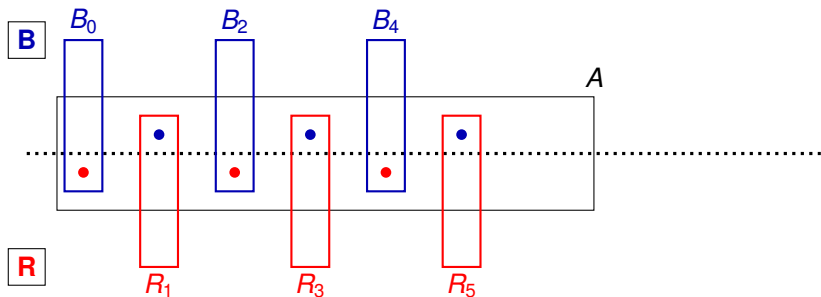
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Theorem (Komjáth)

There is an almost disjoint family $\mathcal{A} \subset [\mathbb{R}]^\omega$ which refines $[\mathbb{R}]^{\omega_1}$ (i.e. for all $X \in [\mathbb{R}]^{\omega_1}$ there is $A \in \mathcal{A}$ with $A \subset X$), and so $\chi(\mathcal{A}) = 2^\omega$.

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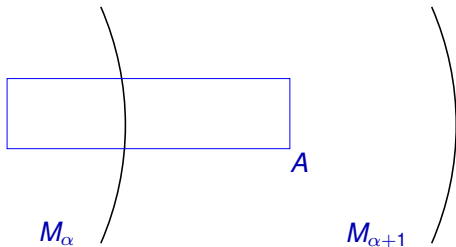
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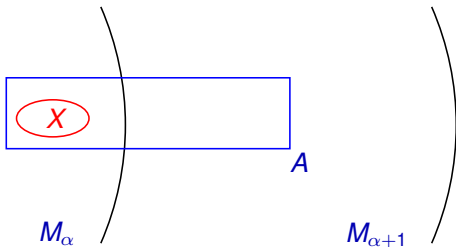
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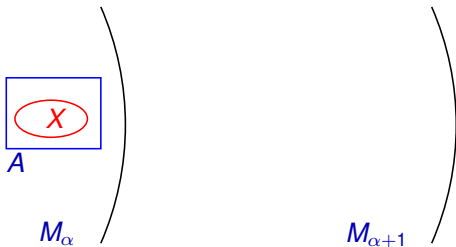
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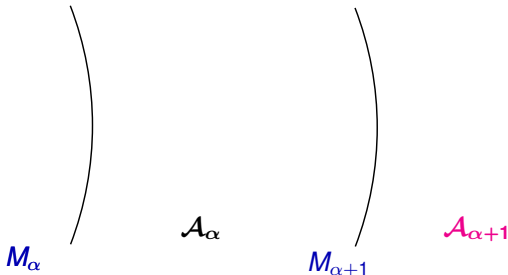


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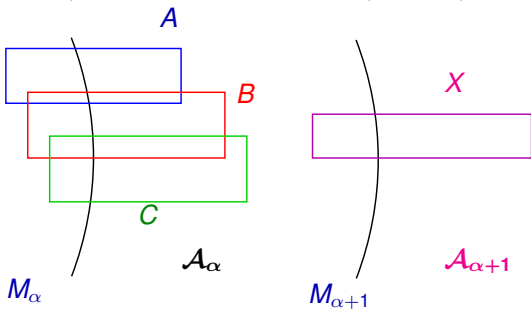
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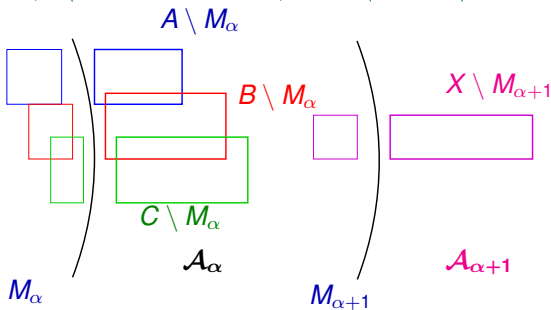
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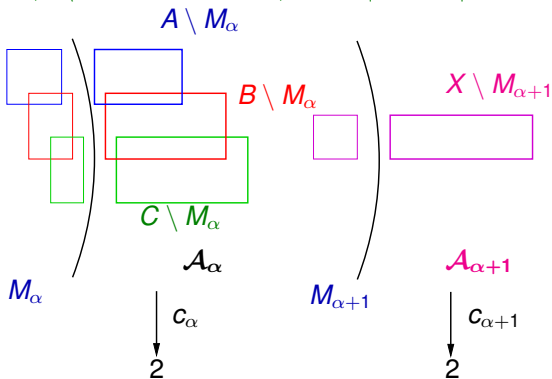
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