Monochromatic paths and path squares in infinite graphs

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The beginning: an infinite result

Theorem (Erdős, Rado, (published in 1987))

Let $r \in \omega$. Suppose that the edges of the countable complete graph K_ω is coloured with r colors. Then there are r disjoint monochromatic paths with different colours which cover all vertices of K_ω.

$$K_\omega \sqsubset (Path_0, \ldots, Path_{r-1})$$

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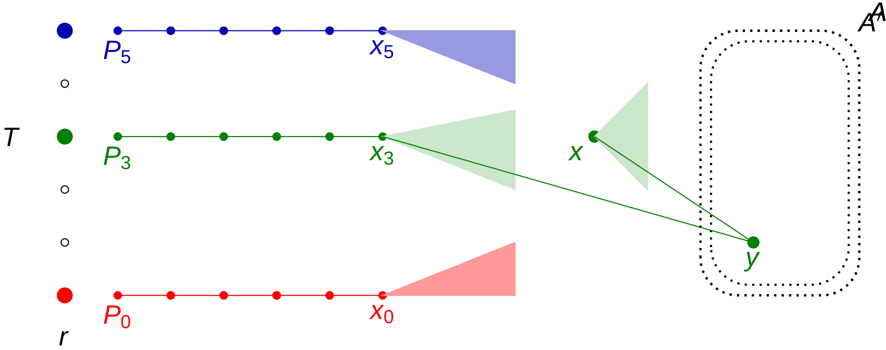


- Proof (Rado): $c : [\omega]^2 \to r$
- $T \subset r$ is perfect iff $\exists$ disjoint finite paths $\{P_t = \ldots x_t : t \in T\}$ with $c[P_t] = \{t\}$ and $\exists A \in [\omega]^\omega$ such that $c(x_t, y) = t$ for all $y \in A$.
- Let T be a maximal perfect set.
- The vertices of K_ω can be partitioned into monochromatic disjoint paths $\{P'_t : t \in T\}$ s.t. $c[P'_t] = \{t\}$.

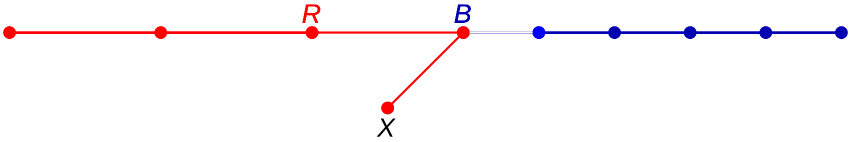
Prelude

- $K_\omega \sqsubset (Path_0, \ldots, Path_{r-1})$

Theorem (Gerencsér, Gyárfás, 1967)

Suppose that the edges of a finite complete graph K_n is coloured with 2 colors. Then there are 2 disjoint monochromatic paths with different colours which cover all vertices of K_n.

$$K_n \sqsubset (Path_0, Path_1)$$



More colors? Cycles instead paths?

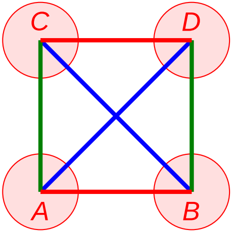
Covers of finite graphs by monochromatic paths and cycles

- $K_\omega \sqsubset (Path_0, \ldots, Path_{r-1})$
- $K_n \sqsubset (Path_0, Path_1)$

Theorem (Kathy Heinrich)

Some 3-edge-coloured K_n can not be covered by disjoint monochromatic paths of different colours.

$$K_n \not\sqsubset (Path_0, Path_1, Path_2)$$



$$|A| = |B| = |C| = |D| = n$$

Conjectures: Every r-edge-coloured K_n can be covered with r vertex-disjoint monochromatic

- paths (Gyárfás, 1989): $K_n \sqsubset (Path)_r$
- cycles (Erdős, Gyárfás, Pyber; Lehel for $r = 2$): $K_n \sqsubset (Cycle)_r$

Theorems:

Covers of infinite graphs

(Erdős, Rado) Let $r \in \omega$. Suppose that the edges of the countable complete graph K_ω is coloured with r colors. Then there are r disjoint, finite or one-way infinite monochromatic paths with different colours which cover all vertices of K_ω.

$$K_\omega \sqsubset (Path_0, Path_1, \ldots, Path_{r-1})$$

Theorem

Let $r \in \omega$. Suppose that the edges of the countable complete graph K_ω is coloured with r colors. Then there are r disjoint, monochromatic two-way infinite paths and cycles with different colours which cover all vertices of K_ω.

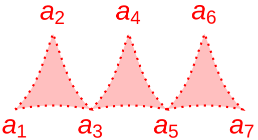
$$K_\omega \sqsubset (Cycle_0, Cycle_1, \ldots, Cycle_{r-1})$$

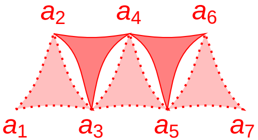
Need: ultrafilter argument

Covers of infinite hypergraphs

Definition

A loose path *in a k-uniform hypergraph is a sequence of edges, $e_1, e_2, \ldots$ such that for $|e_i \cap e_{i+1}| = 1$ and $e_i \cap e_j = \emptyset$ for $i + 1 < j$.*





A tight path *in a k-uniform hypergraph is a sequence of distinct vertices such that every consecutive set of k vertices forms an edge.*

Theorem (Gyárfás, G. N. Sárközy, 2012)

Given any r-edge colouring of K_ω^ℓ the vertex set can be partitioned into monochromatic loose paths of distinct colors.

Theorem (M. Elekes, D. Soukup, -, Z. Szentmiklóssy)

Given any r-edge colouring of K_ω^ℓ the vertex set can be partitioned into monochromatic tight paths of distinct colors.

Covers by power of paths

Definition

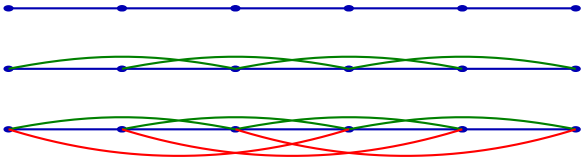
Suppose that G is a graph and $k \in \omega$. The k^{th} power of G is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff $dist_G(v, w) \leq k$.

What is a power of a path?

P_6

P_6^2

P_6^3



Theorem (M. Elekes, D. Soukup, -, Z. Szentmiklóssy)

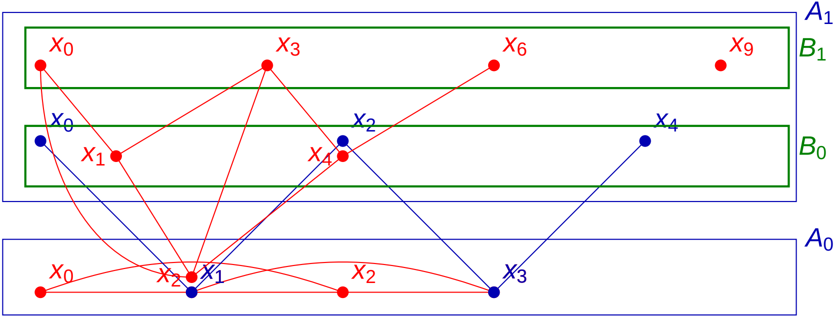
Let $k, r \in \omega$. Suppose that c is a colouring of the edges K_ω with r colours. Then the vertices can be partitioned into $\leq r^{(k-1)r+1}$ infinite monochromatic k^{th} powers of paths and a finite set.

For $k = r = 2$ we have a partition into ***5*** *monochromatic squares of paths.*

$K_\omega \sqsubset (PathSquare)_{2,5}$

- $c : [\omega]^2 \to 2$
- $\mathcal{U}$ ultrafilter on ω
- $A_0 \in \mathcal{U}$
- $\mathcal{V}$ ultrafilter on A_1
- $B_0 \in \mathcal{U}$
- $B_1 \sqsubset PathSquare_1$

- $N_G(x, i) = \{y : c(x, y) = i\}$
- $A_i = \{x \in \omega : N_G(x, i) \in \mathcal{U}\}$
- $A_0 \sqsubset PathSquare_0$ • $A_1 \sqsubset Path_1$
- $B_i = \{x \in A_1 : N_G(x, i) \cap A_1 \in \mathcal{V}\}$
- $B_0 \sqsubset PathSquare_0$



Covers of uncountable graphs

Definition (Rado)

$P = (V, E)$ is a path iff there is a well ordering $\preceq$ on V such that any two vertices is connected by a $\preceq$-monotone (finite) path.

$\{p_\alpha : \alpha < \delta\}$ is a path iff

- $\{p_\alpha, p_{\alpha+1}\} \in E$ for $\alpha + 1 < \delta$
- $\{\alpha < \beta : \{p_\alpha, p_\beta\} \in E\}$ is cofinal in β for all limit $\beta < \delta$

Theorem (M. Elekes, D.Soukup, -, Z. Szentmiklóssy)

Given any 2-edge colouring of K_{ω_1} we can partition the vertices into two monochromatic paths of different colors.

$$K_{\omega_1} \sqsubset (Path_0, Path_1).$$

Theorem (D. Soukup)

If G is an infinite complete graph and $r \in \omega$, then for every r-edge colouring of G we can partition the vertices into finitely many monochromatic paths.

$$K_\kappa \sqsubset (Path)_{r,<\omega}.$$

Problems

- $K_\omega^\ell \sqsubset (TightPath_0, \ldots, TightPath_{r-1})$.
- Problem: $K_\omega^\ell \sqsubset (TightCycle_0, \ldots, TightCyle_{r-1})$??
- $K_\kappa \sqsubset (Path)_{r,<\omega}$.
- Problem: $K_\kappa \sqsubset (Path)_{r,f(r)}$.
- $K_\omega \sqsubset^* (k^{\mathrm{th}}\text{-}PowerofPath)_{r,r^{(k-1)r+1}}$
- ⋆ Problem $K_\omega \sqsubset (k^{\mathrm{th}}\text{-}PowerofPath)_{r,g(k,r)}$
- $K_\omega \sqsubset (PathSquare)_{2,5}$
- Problem $K_\omega \sqsubset (PathSquare)_{2,3}$

Infinitely many colors????

Thank you!