Cardinal sequences of scattered spaces

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Introduction

A space X is **scattered** iff every non-empty subspace Y has an **isolated point**:

$I(Y) = \{p \in Y : p$ **is isolated in** $Y\} \neq \emptyset$.

Cantor-Bendixson Theorem

Every topological space is the disjoint union of a **crowded closed subspace** P**,** and a **scattered open subspace** X .

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Cantor-Bendixson Hierarchy

$I(Y) = \{p \in Y : p$ is isolated in $Y\}$
X is **scattered** iff $I(Y) \neq \emptyset$ for each nonempty $Y \subset X$.

- $I_0(X)$ is the isolated points of X
- $I_1(X)$ is the isolated points of $X \setminus I_0(X)$

The β^{th} **Cantor-Bendixson level** of X is

$$I_\beta(X) = I(X \setminus \cup\{I_\alpha(X) : \alpha < \beta\})$$

T.F.A.E:

- X is **scattered**
- $X = \cup\{I_\alpha(X) : \alpha \in On\}$.
- X is **right-separated** (there is a **well-ordering** $\preceq$ of X such that $\{y \in X : y \preceq x\}$ is open in X for each $x \in X$.)

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Invariants of scattered spaces

The ***height*** of X:

$$\mathrm{ht}(X) = \min\{\beta : I_\beta(X) = \emptyset\}.$$

The **width** of X:

$$wd(X) = \sup\{|I_\alpha(X)| : \alpha < \mathrm{ht}(X)\}$$

The ***reduced height***:

$$\mathrm{ht}^-(X) = \min\{\alpha : I_\alpha(X) \text{ is finite}\}.$$

Clearly, one has

$$\mathrm{ht}^-(X) \le \mathrm{ht}(X) \le \mathrm{ht}^-(X) + 1.$$

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Main question

What are the cardinal sequences of (locally) compact scattered spaces (or: superatomic boolean algebras)?

$\mathcal{C}(\alpha) = \{SEQ(X) : X$ **locally compact scattered,** $\mathrm{ht}^-(X) = \alpha\}$.

Characterize $\mathcal{C}(\alpha)$**!**

Decide whether $s = SEQ(X)$ **for some LCS space** X

LCS space $\equiv$ **locally compact scattered space**

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$\langle\kappa\rangle_\delta$: constant κ sequence of length δ

Telgarsky, 1968: $\langle\omega\rangle_{\omega_1} \in \mathcal{C}(\omega_1)$**?**

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- Roitman: Con(ZFC + $\langle\omega\rangle_{\omega_1} ^\frown \langle\omega_2\rangle \in \mathcal{C}(\omega_1 + 1)$)
- Baumgartner-Shelah: Con(ZFC + $\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$)
- Martinez: Con(ZFC + $\forall\delta < \omega_3\ \langle\omega\rangle_\delta \in \mathcal{C}(\delta)$.)

Cones and the meet function

X is an **LCS space**

$\{I_\alpha(X) : \alpha < \mathrm{ht}(X)\}$

Fact: X is 0-dimensional

So for each $x \in I_\alpha(X)$ we can fix compact open $U(x) \ni x$ such that $U(x) \setminus \{x\} \subset I_{<\alpha}(X)$

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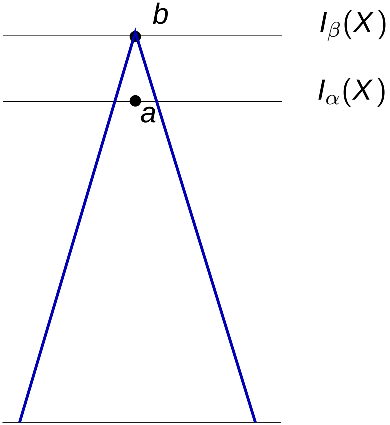

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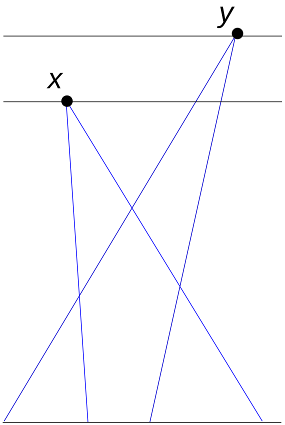

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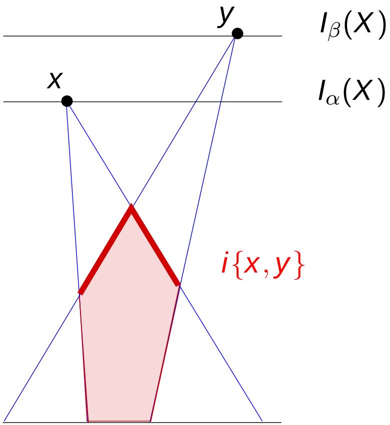

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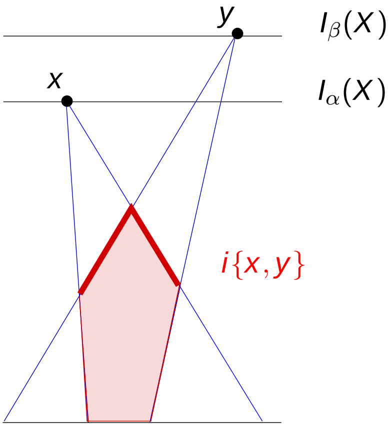

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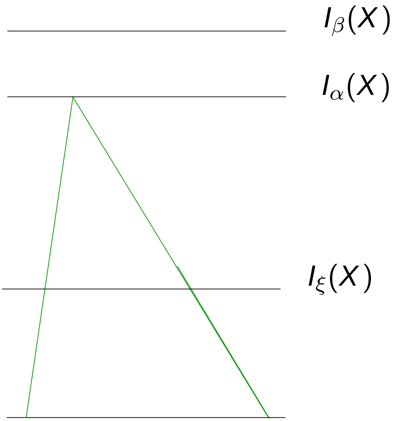

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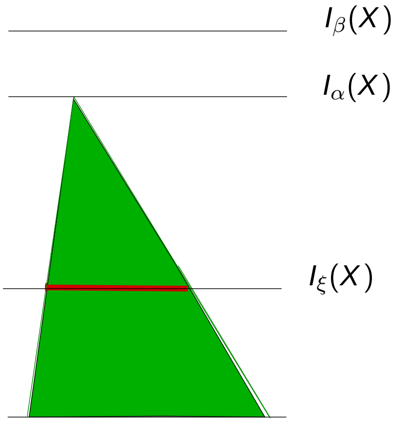

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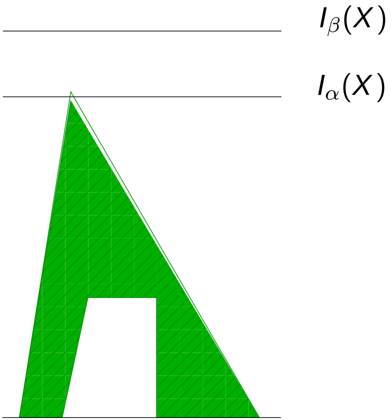

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(0) if $x \in X_\alpha$, $y \in X_\beta$, $x \prec y$ then $\alpha < \beta$.

(1) if x and y are $\preceq$-incomparable elements of X then $t \preceq x \wedge t \preceq y$ iff $\exists s \in i\{x, y\}$ $t \preceq s$.

(2) if $x \in X_\alpha$ and $\xi < \alpha$ then there are infinitely many $y \in X_\xi$ with $y \preceq x$.

Let $U(x) = \{x' \in X : x' \preceq x\}$. Then

- $\{U(x), X \setminus U(x) : x \in X\}$ is a subbase of an LCS space of $\langle X, \tau \rangle$,
- $U(x)$ is compact open,
- $I_\alpha(X, \tau) = X_\alpha$.

Construction strategy

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- Construct an LCS space with cardinal sequence $\langle \kappa_\alpha : \alpha < \mu \rangle$.
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for each family $\mathcal{A} \subset [\lambda]^{<\kappa}$ of pairwise disjoint sets with $|\mathcal{A}| = \kappa^+$

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Theorem (Galvin, Roitman)

It is consistent that there is a ω_1-strongly unbounded function on ω_2.

Theorem (Koszmider)

If $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+ \leq \lambda$, then there is a κ^+- strongly unbounded function on λ and $\kappa = \kappa^{<\kappa}$ in some cardinal-preserving generic extension of the ground model.

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It is consistent that there is a ω_1-strongly unbounded function on ω_2.

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If $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+ \leq \lambda$, then there is a κ^+- strongly unbounded function on λ and $\kappa = \kappa^{<\kappa}$ in some cardinal-preserving generic extension of the ground model.

Beyond the width ω_2

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for each family $\mathcal{A} \subset [\lambda]^{<\kappa}$ of pairwise disjoint sets with $|\mathcal{A}| = \kappa^+$

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- $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle \in \mathcal{C}(\omega_1+1) \overset{?}{\Longleftrightarrow} \langle\omega\rangle_{\delta} {}^\frown \langle\omega_2\rangle \in \mathcal{C}(\omega_1+1)$
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- Is there a c c poset P s.t. $V^P \models \langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$?

- Observation: $\langle\omega\rangle_{\omega_2} {}^\frown \langle\omega_1\rangle \notin \mathcal{C}(\omega_2+1)$
- Characterize $\mathcal{C}(\delta)$ for $\omega_2 \leq \delta < \omega_3$!

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In the Cohen model $\langle\omega\rangle_{\omega_2} \notin \mathcal{C}(\omega_2)$.

Theorem (Juhász, Shelah, S, Szentmiklóssy)

$\{\alpha : |I_\alpha(X)| = \omega\} \leq \omega_1$ *in the Cohen model.*

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$\{\alpha : |I_\alpha(X)| = \omega\} \leq \omega_2$.

Shelah: $\boxed{2^{\aleph_0} < \aleph_\omega \Longrightarrow (\aleph_\omega)^{\aleph_0} < \aleph_{\omega_4}}$

If the Conjecture holds then $2^{\aleph_0} < \aleph_\omega \Longrightarrow (\aleph_\omega)^{\aleph_0} < \aleph_{\omega_3}$

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